Analysis of a Tubular Gas Lens

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If a cool gas is blown into a hot tube, it acts as a positive lens which will focus a light beam passing through the tube.

Using a theory presented in Ref. 3, we give curves which show the temperature distribution in the tube as a function of the distance from the tube axis and also as a function of the distance along the axis.

The focusing power of the lens is described by the difference in phase angle between a ray on the tube axis minus a ray at arbitrary distances from this axis and also as the second derivative of the phase angle on the axis of the tube. The phase curves, as a function of distance r from the tube axis, follow very closely an r^2 dependence. Expressions are given for the focal length of the lens.

The power consumption of the lens is discussed, and a figure of merit is defined as focusing power per watt. The gas used for this lens should be selected such that $(n - 1)/k$ is as large as possible (where n is the index of refraction, k the heat conductivity of the gas).

Using CO_2 and a $\frac{1}{4}$-inch ID tube 5 inches long heated 20°C above the incoming gas, a focal length of 5 feet with a power consumption of 0.325 watt is calculated; the focal length is inversely proportional to power consumption within certain limits.

I. INTRODUCTION

A communications system using light as the carrier of intelligence needs an efficient medium to propagate light from transmitter to receiver. Among the several alternatives, the idea of Goubau and Schwering[1] of confining and propagating an electromagnetic wave with a system of lenses appears promising. However, in a lens-waveguide system there is a wide range of possibilities as to what types of lenses to use. Conventional glass lenses present problems, since they may not only absorb light in the glass medium itself, but furthermore present important reflection losses which can be only partially avoided by special techniques such as coating the lens surfaces or making use of the Brewster angle.

Even if such corrective measures are used, there is still residual reflection and scattering of light due to unavoidable surface irregularities.

It appears that most of the problems connected with glass lenses could be overcome if, instead of a high-index medium such as glass, a very low-index focusing medium were used. If the transition from air into a dense medium could be avoided, the problem of light reflection would not exist. Gases present themselves as an obvious choice of a low-index dielectric medium. Their dielectric constant can be influenced by changing their density. A change of density is most easily effected by varying the gas temperature.

D. W. Berreman[2] built a successful gas lens by maintaining a temperature gradient between a hot helix and a cold cylindrical enclosure. Alternatively, D. W. Berreman and S. E. Miller proposed a gas lens formed by blowing a cool gas into a hot tube (Fig. 1). Since the gas heats up first at the wall of the tube and remains cool longer at its center, it has a density distribution of higher-density gas in the center of the tube and decreasing density towards the wall. Since an increase in density is accompanied by an increase in dielectric constant, it is easy to understand that the cool gas flowing through the hot tube acts as a positive lens and tends to focus a light beam traveling along the axis of the tube.

We present in this article some theoretical results of the temperature distribution in the gas and the difference in phase angle between two light beams, one traveling along the tube axis and the other traveling closer to the wall of the tube. This phase difference is a measure of the focusing power of the lens. For an economical lens we want maximum phase shift with a minimum of thermal power. We present curves showing the power consumption of the lens as well as the ratio of phase difference to power consumption. These data allow the construction of an optimum lens. Different gases give different lens properties. A gas is most efficient if the ratio $(n - 1)/k$ is large, where n is the index of refraction of the gas and k is its heat conductivity.

II. TEMPERATURE DISTRIBUTION

The theory of temperature distribution in a cool gas which is blown into a hot tube of constant temperature is presented in Ref. 3.

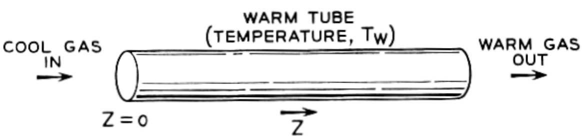

Fig. 1 — Gas lens using forced flow in a tube.

It is assumed that the gas flow is laminar and has the radial velocity distribution of a viscous fluid

$$v(r) = v_0[1 - (r/a)^2] \tag{1}$$

where

r = distance from tube axis
a = radius of tube
v_0 = gas velocity at $r = 0$.

The temperature T of the gas is given as

$$\theta = T_w - T$$

where T_w is the temperature of the wall of the tube. It is normalized with respect to

$$\theta_0 = T_w - T_0$$

with T_0 being the temperature of the cool gas before it enters the hot tube. θ/θ_0 is expanded in terms of functions $R_n(r/a)$, which are shown in Fig. 2 for $n = 0$, 1 and 2. Values of $R_n(r/a)$ are listed in Table I. The temperature depends on the distance z measured from the beginning of the hot tube, the gas velocity v_0, and the following material parameters

k = heat conductivity measured in cal/cm sec deg)
(deg = degrees Kelvin)
ρ = gas density in gram/cm^3
c_p = specific heat at constant pressure in cal/gram.

All these parameters depend somewhat on the temperature but are considered constant in the derivation of the theory. They enter the equations in the combination

$$\sigma = k/av_0\rho c_p\,. \tag{2}$$

Table I — R Functions of Fig. 2

x	$R_0(x)$	$F(x)$	$R_1(x)$	$R_2(x)$
0	1	1	1	1
0.1	0.9819	0.9805	0.8923	0.753
0.2	0.9290	0.9261	0.6067	0.206
0.3	0.8456	0.8432	0.2367	−0.290
0.4	0.7382	0.7382	−0.1062	−0.407
0.5	0.6147	0.6175	−0.3399	−0.204
0.6	0.4833	0.4880	−0.4317	0.104
0.7	0.3506	0.3535	−0.3985	0.278
0.8	0.2244	0.2244	−0.3051	0.278
0.9	0.1069	0.1041	−0.1637	0.144
1.0	0	0	0	0

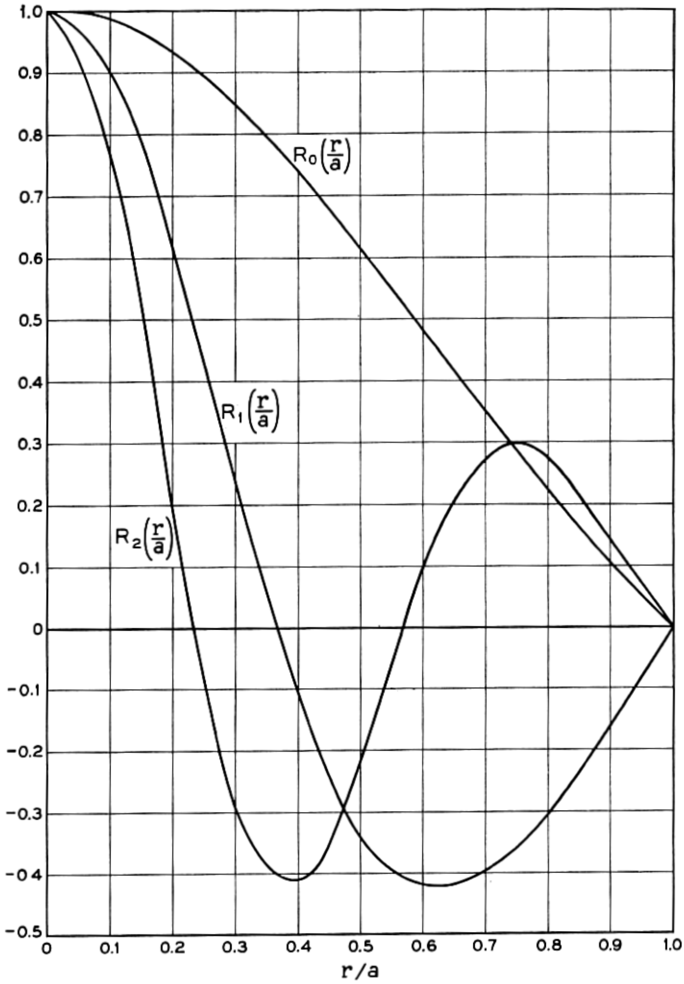

Fig. 2 — Functions R_0, R_1 and R_2 vs r/a.

The first three terms of the infinite series describing the temperature distribution in the tube are

$$\frac{\theta}{\theta_0} = 1.477 \exp\left(-\ 7.316\sigma \frac{z}{a}\right) R_0\left(\frac{r}{a}\right) - 0.810 \exp\left(-44.36\sigma \frac{z}{a}\right) \cdot R_1\left(\frac{r}{a}\right) + 0.385 \exp\left(-106\sigma \frac{z}{a}\right) R_2\left(\frac{r}{a}\right) \pm \cdots. \tag{3}$$

The approximation is fairly poor at $z = 0$. However, the exponential factors in the higher terms of the series drop off very rapidly as z increases so that the approximation is already very good for values of

$$\sigma(z/a) > 0.01.$$

Fig. 3 shows θ/θ_0 at $r = 0$ as a function of $\sigma(z/a)$. It is apparent that for $\sigma(z/a) > 0.05$, the distribution of θ/θ_0 drops off exponentially. Fig. 4 shows the r/a dependence of θ/θ_0 for different values of $\sigma(z/a)$.

To make Figs. 3 and 4 more meaningful, we list in Table II the material parameters for several gases at 20°C and a pressure of 760 mm Hg.

III. POWER CONSUMPTION

The principle of operation of our gas flow lens requires that we heat the cool gas inside the hot tube. Even if we neglect all power losses to the environment, we have to spend a certain amount of heat power to operate our lens. For a subsequent study of lens efficiency we need to know this basic power consumption. At any given length z of the tube we obtain the power absorbed by the gas as

$$\begin{aligned} P(z) &= \int_0^a [T(r,z) - T_0]\rho c_p v(r) 2\pi r \, dr \\ &= 2\pi\rho c_p v_0 \theta_0 \int_0^a r\left[1 - \left(\frac{r}{a}\right)^2\right]\left[1 - \frac{\theta(r,z)}{\theta_0}\right] dr. \end{aligned} \tag{4}$$

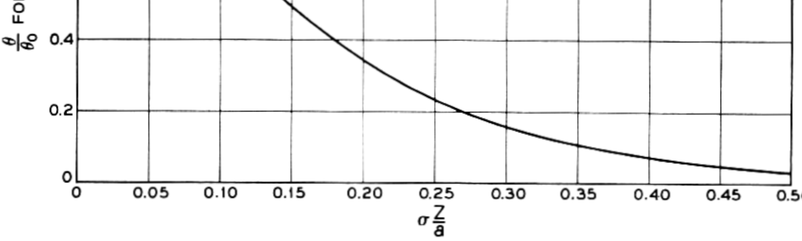

Fig. 3 — Normalized gas temperature on tube axis vs normalized distance along tube.

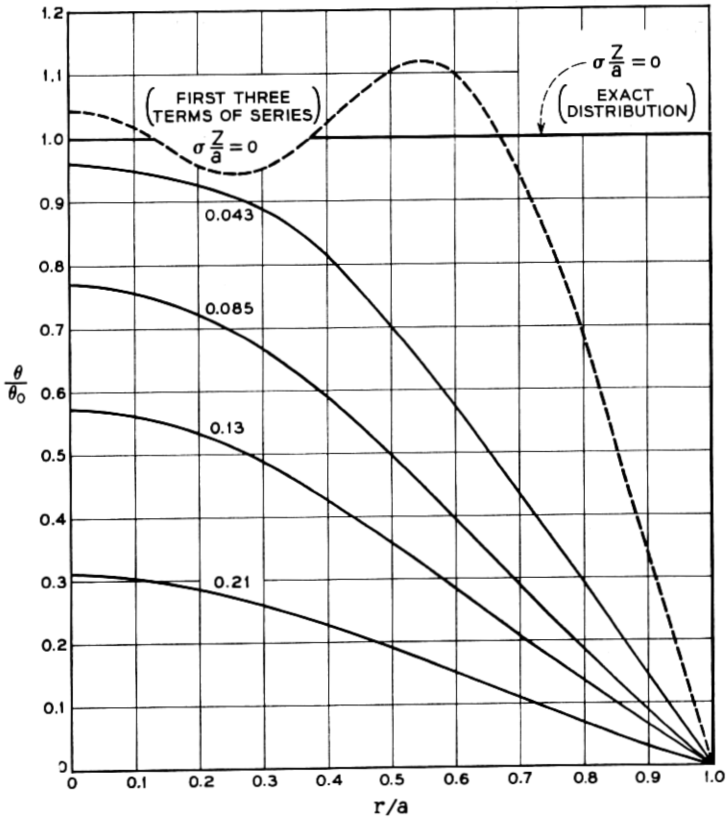

Fig. 4 — Normalized gas temperature vs radial position with longitudinal position as a parameter.

In order to be able to perform the integration easily, we restrict ourselves to values of

$$\sigma(z/a) > 0.05$$

which allows us to express θ/θ_0 by the first term of (3). In addition, we replace $R_0(x)$ by

$$R_0(x) \cong F(x) = 1 - 2.06\, x^2 + 1.06\, x^3. \tag{5}$$

This approximation deviates no more than 2.5 percent from the actual value of $R_0(x)$. The values of $F(x)$ can be compared to those of $R_0(x)$ in Table I. The integration can now be performed easily, and we obtain:

$$P = \frac{\pi}{2} a^2 \rho c_p v_0 \theta_0 \left[1 - 0.820 \exp\left(-7.316\sigma \frac{z}{a} \right) \right]. \tag{6}$$

TABLE II — GAS PARAMETERS VS TEMPERATURE

Gas	k (cal/cm sec deg)	ρ (gram/cm³)	c_p (cal/gram deg)	$av_0\sigma = k/\rho c_p$ (cm²/sec)	$n - 1$	$(n-1)/k$ (cm sec deg/cal)
CO_2	$3.93\ 10^{-5}$	$1.84\ 10^{-3}$	0.199	0.107	$4.20\ 10^{-4}$	10.7
NH_3	$5.90\ 10^{-5}$	$0.72\ 10^{-3}$	0.523	0.157	$3.48\ 10^{-4}$	5.9
CH_4	$7.80\ 10^{-5}$	$0.67\ 10^{-3}$	0.528	0.220	$4.13\ 10^{-4}$	5.3
Air	$6.28\ 10^{-5}$	$1.21\ 10^{-3}$	0.240	0.216	$2.73\ 10^{-4}$	4.35
H_2	$41.0\ 10^{-5}$	$0.084\ 10^{-3}$	3.39	1.44	$1.23\ 10^{-4}$	0.30
He	$35.0\ 10^{-5}$	$0.166\ 10^{-3}$	1.25	1.69	$0.34\ 10^{-4}$	0.097

Using ρ and c_p from Table II, P is in calories/sec. Fig. 5(a) shows the power consumption as a function of normalized lens length, $\sigma z/a$. An alternative form of (6) brings out the dependence of P on the flow velocity v_0 more clearly:

$$P = \frac{\pi}{2} kz\theta_0 \frac{v_0}{V}\left[1 - 0.820 \exp\left(-7.316 \frac{V}{v_0}\right)\right]. \tag{7}$$

The quantity

$$V = kz/a^2\rho c_p = \sigma(z/a)v_0 \tag{8}$$

has the dimension of velocity and is characteristic of the gas and the tube geometry. Fig. 5(b) shows the power consumption as a function of normalized gas velocity, v_0/V.

The ratio of V/v_0 can be related to the time $t_0 = z/v_0$ which it takes the gas particles on the axis to traverse the tube of length z with the velocity v_0 and to a time τ which is defined by

$$\frac{1}{\tau} = \frac{\dfrac{dT(0,t)}{dt}}{T_w - T(0,t)}. \tag{9}$$

τ is characteristic of the heat diffusion rate in a gas which rests in a tube whose wall temperature is T_w. At $t = 0$ the gas has the uniform temperature $T_0 < T_w$. Its temperature at a given radius r and time t is $T(r,t)$, so that $T(0,t)$ is the gas temperature at the tube axis at time t. $1/\tau$ is the time rate of temperature rise on the axis per degree of temperature difference between wall and axis.

It is shown in Appendix A that

$$V/v_0 = \sigma z/a = 0.173(t_0/\tau). \tag{10}$$

This shows that V/v_0 expresses the ratio of the time it takes the gas particles (on the tube axis) to flow through the tube of length z to the heat diffusion rate on the tube axis. Equation (10) may be used to replace

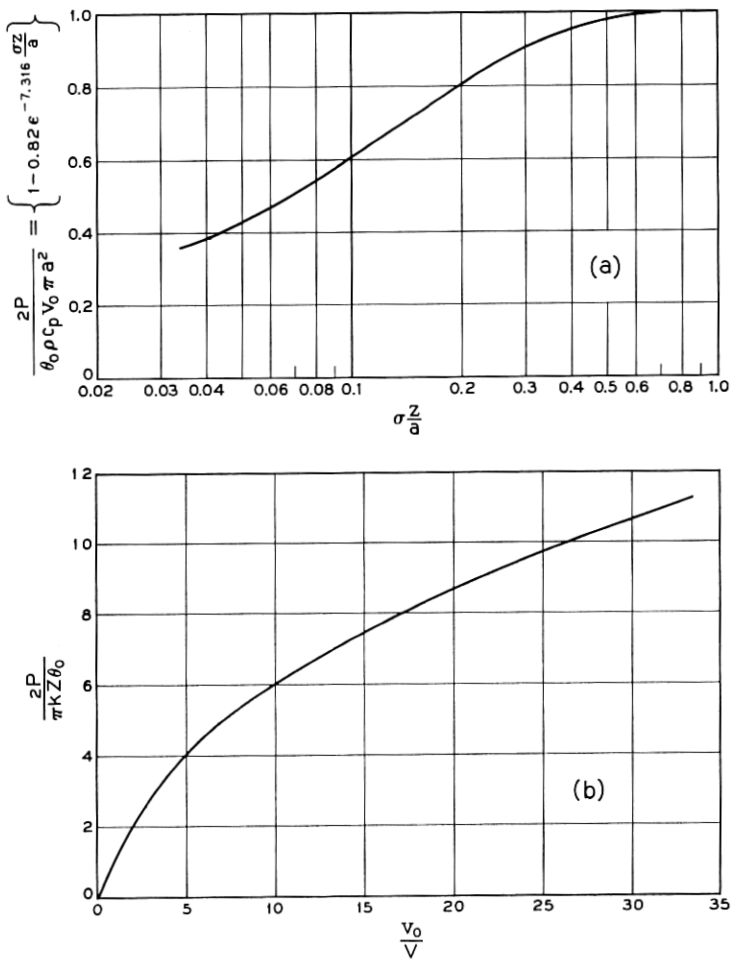

Fig. 5 — (a) Normalized power flow to the gas vs longitudinal position. (b) Normalized power flow to the gas vs normalized gas velocity.

V/v_0 in (7) and in the other places where V/v_0 or $\sigma z/a$ appears, resulting (for example) in

$$\exp\left(-7.316\,\frac{\sigma z}{a}\right) = \exp\left(-7.316\,\frac{V}{v_0}\right) = \exp\left(-1.265\,\frac{t_0}{\tau}\right).$$

The parameters lens length z and gas velocity v_0 are important variables for other reasons, and the first two forms of the exponential may be preferred. We may note, however,

$$\tau = 0.173(a^2 \rho c_p/k)$$

and typical values are 0.161 second and 0.08 second for CO_2 and air, respectively, when $a = 0.125$ inch. It is surprising that this time constant is so short.

Another physical meaning one can give V is that it represents that velocity of gas flow along the pipe axis which assures that θ/θ_0 drops from its initial value of one at the beginning of the tube to

$$\theta/\theta_0 = 9.910^{-4} \approx 10^{-3}$$

on the axis at its end.

IV. FOCUSING ACTION

A lens focuses because the optical path length varies for rays traveling at different distances from its axis.

We describe the focusing action of our lens by the phase angle of a ray traveling parallel to the axis of the structure. The phase angle is given by

$$\Phi(r,z) = \beta_0 \int_0^z n(r,x)\, dx. \tag{11}$$

Here, $\beta_0 = 2\pi/\lambda_0$ is the free-space propagation constant of the light beam, and n is the index of refraction of the gas.

$$n(r,x) = 1 + (n_0 - 1) \frac{T_0}{T(r,x)} \tag{12}$$

$$\frac{T_0}{T} = \frac{1}{1 + \frac{\theta_0}{T_0}\left(1 - \frac{\theta}{\theta_0}\right)} \approx 1 - \frac{\theta_0}{T_0}\left(1 - \frac{\theta}{\theta_0}\right). \tag{13}$$

The last step is an approximation for $\theta_0/T_0 \ll 1$. The temperature in (12) has to be expressed in degrees Kelvin.

We decompose $\Phi(r,z)$ into two parts:

$$\Phi(r,z) = \varphi + \psi(r,z). \tag{14}$$

The first part

$$\varphi = \beta_0 \left[1 + (n_0 - 1)\left(1 - \frac{\theta_0}{T_0}\right)\right] z \tag{15}$$

is independent of the position r of the ray in the gas lens, while the second part

$$\psi = \beta_0(n_0 - 1)\frac{\theta_0}{T_0}\int_0^z \frac{\theta(r,x)}{\theta_0}\,dx$$

$$= \beta_0 z(n_0 - 1)\frac{\theta_0}{T_0}\cdot\frac{v_0}{V}\left\{0.202R_0\left(\frac{r}{a}\right)\left[1 - \exp\left(-7.316\frac{V}{v_0}\right)\right]\right. \tag{16}$$
$$- 0.0183R_1\left(\frac{r}{a}\right)\left[1 - \exp\left(-44.3\frac{V}{v_0}\right)\right]$$
$$\left. + 0.00363R_2\left(\frac{r}{a}\right)\left[1 - \exp\left(-106\frac{V}{v_0}\right)\right] + \cdots\right\}$$

accounts for the different amounts of phase shift in different parts of the lens.

The difference between the phase angle of a ray traveling along the lens axis and the phase angle of a ray traveling at a distance r from the axis is

$$\Delta\Phi = \beta_0 z(n_0 - 1)\frac{\theta_0}{T_0}\frac{v_0}{V}\left\{0.202(1 - R_0)\left[1 - \exp\left(-7.316\frac{V}{v_0}\right)\right]\right.$$
$$- 0.0183(1 - R_1)\left[1 - \exp\left(-44.3\frac{V}{v_0}\right)\right] \tag{17}$$
$$\left. + 0.00363(1 - R_2)\left[1 - \exp\left(-106\frac{V}{v_0}\right)\right] + \cdots\right\}.$$

This form of $\Delta\Phi$ shows clearly its dependence on flow velocity for a fixed tube length z. To study the dependence of $\Delta\Phi$ for fixed flow rate and varying length, the following form is preferable.

$$\Delta\Phi = \beta_0(a/\sigma)(n_0 - 1)(\theta_0/T_0)\{0.202(1 - R_0)$$
$$\cdot(1 - \exp(-7.316\sigma z/a)) - 0.0183(1 - R_1)(1 - \exp(-44.3\sigma z/a)) \tag{18}$$
$$+ 0.00363(1 - R_2)(1 - \exp(-106\sigma z/a)) + \cdots\}.$$

Fig. 6 shows a plot of $\Delta\Phi$ versus length of lens for $r/a = 0.4$. That means that we compare the phase difference between a ray on the axis and another at distance $r = 0.4a$ from the axis of the lens.

Fig. 7 shows the phase difference at a fixed length z as a function of gas velocity. In this case, $\Delta\Phi$ goes through a maximum which for $r/a = 0.4$ occurs at $v_0/V = 6.9$. The position of this maximum depends somewhat on the radius r of the ray used for phase comparison with the axial ray. Appendix B gives the theory and Table III gives the values of

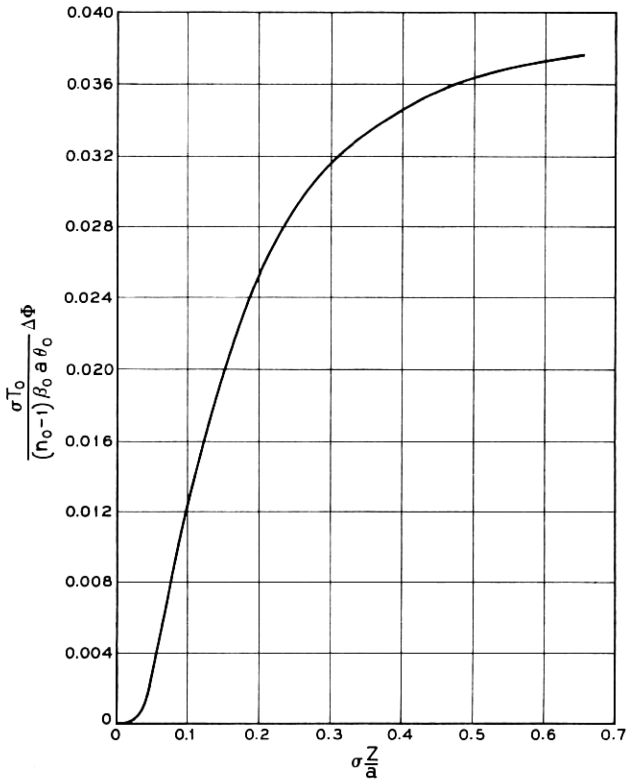

Fig. 6 — Phase difference between ray at $r = 0$ and ray at $r = 0.4a$ vs longitudinal position.

the position of the maximum v_0/V for different values of r/a. Table III shows that the position of the maximum does not change much with the radial position of the reference ray.

We can explain physically this maximum in $\Delta\Phi$ versus v_0 as follows. At very low gas velocity the majority of the gas in the tube is at the same temperature — the temperature of the walls, T_w. It is heated up in a time τ [see (9)] after entering. With little temperature difference between the gas at $r = 0$ and at $r > 0$ there is little $\Delta\Phi$. As v_0 increases, the gas on the axis remains at or near T_0, but that nearer the walls is heated because it flows more slowly [see (1)] and larger $\Delta\Phi$ develops. Beyond some velocity, further increases in velocity cause the gas at $r = 0.4$a (for example) to leave the tube at lower and lower temperatures — i.e., less temperature difference will exist between $r = 0$ and $r = 0.4a$ because the

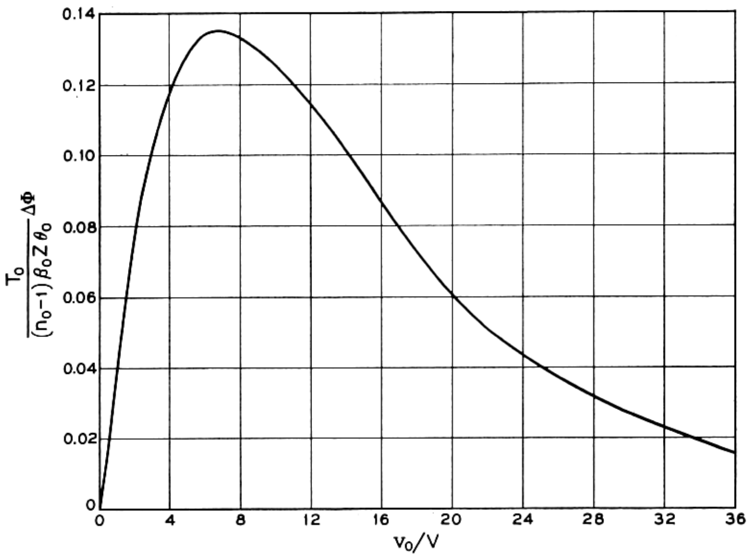

Fig. 7 — Phase difference between ray at $r = 0$ and ray at $r = 0.4a$ vs gas velocity.

transit time of the gas through the tube becomes less than the thermal diffusion time constant τ. Thus, at high velocities the $\Delta\Phi$ decreases. Note that this explanation (and the theory) depends upon laminar flow of the gas. If there is radial mixing of the gas, less $\Delta\Phi$ would be expected than predicted above, and the maximum in $\Delta\Phi$ versus v_0 might not occur.

The second derivative $d^2\Phi/d(r/a)^2$ is a good measure of the effectiveness of the lens for light rays close to its axis. For a glass lens

$$\Phi = n\beta_0\, d(r)$$

where $d(r)$ is the thickness of the lens as a function of the distance from its axis. The radius of curvature R of the glass lens is given by

$$\frac{1}{R} = \frac{1}{n\beta_0}\frac{d^2\Phi}{dr^2}.$$

In order to be able to take the second derivative of Φ we have to express the functions R_0, R_1, and R_2 by power series with respect to r/a. For

Table III — Maximum v_0/V for Values of r/a

r/a	0.2	0.4	0.6
v_0/V	6.73	6.9	8.26

the second derivative on the axis at $r = 0$, it is sufficient to know the coefficient of $(r/a)^2$ in the expansion. Jakob[3] gives a series expansion of the R-functions.

$$R_n = 1 - \tfrac{1}{4}\beta_n^2 (r/a)^2 \pm \cdots \tag{19}$$

with $\beta_0 = 2.705$, $\beta_1 = 6.66$, and $\beta_2 = 10.3$.

We get these values

$$\begin{aligned} \left(\frac{d^2\Phi}{d\left(\frac{r}{a}\right)^2} \right)_{r=0} = \beta_0 z(n_0 - 1) \frac{\theta_0}{T_0} \frac{v_0}{V} \Big\{ & 0.738 \left[1 - \exp\left(-7.316 \frac{V}{v_0} \right) \right] \\ & - 0.405 \left[1 - \exp\left(-44.3 \frac{V}{v_0} \right) \right] \\ & + 0.192 \left[1 - \exp\left(-106 \frac{V}{v_0} \right) \right] + \cdots \Big\}. \end{aligned} \tag{20}$$

The maximum of this curve as a function of v_0/V appears at $v_0/V = 6.75$.

Fig. 8(a) is a plot of a normalized value of $d^2\Phi/d(r/a)^2$ as a function of v_0/V, while Fig. 8(b) shows it (with a different normalization) as a function of $\sigma(z/a)$.

The r/a dependence of $\Delta\Phi$ is shown in Fig. 9.

In order to show what values the phase difference might actually assume, and also to compare different gases, we have plotted $\Delta\Phi$ in Fig. 10 for several gases and the following geometry and flow rate:

$2a = 0.25$ inch
$v_0 = 212$ cm/sec, corresponding to 2 liters/minute or 4.77 miles/hr.
$\beta_0 = 1.07 \times 10^5 \text{ cm}^{-1}$, corresponding to $\lambda_0 = 5890$Å
$T_w = 343°$K
$T_0 = 293°$K

and with the values of $n_0 - 1$ and σ as listed in Table II. These curves assume a tube length z so long that no further $\Delta\Phi$ would be realized with a longer z (i.e., outgoing gas at uniform temperature).

It is interesting to compare the r/a dependence of $\Delta\Phi$ to the simple function $c(r/a)^2$. For this purpose we use (17), which is written so that $\Delta\Phi = 0$ at $r = 0$. Fig. 11 gives the normalized value of $\Delta\Phi$ as a function of r/a for several values of the gas velocity v_0/V. For comparison the function $c(r/a)^2$ is shown by dotted lines. The constant c is adjusted so that both curves coincide at $r/a = 0.4$. The actual curves of $\Delta\Phi$ are surprisingly close to the simple square law dependence in all cases. If the

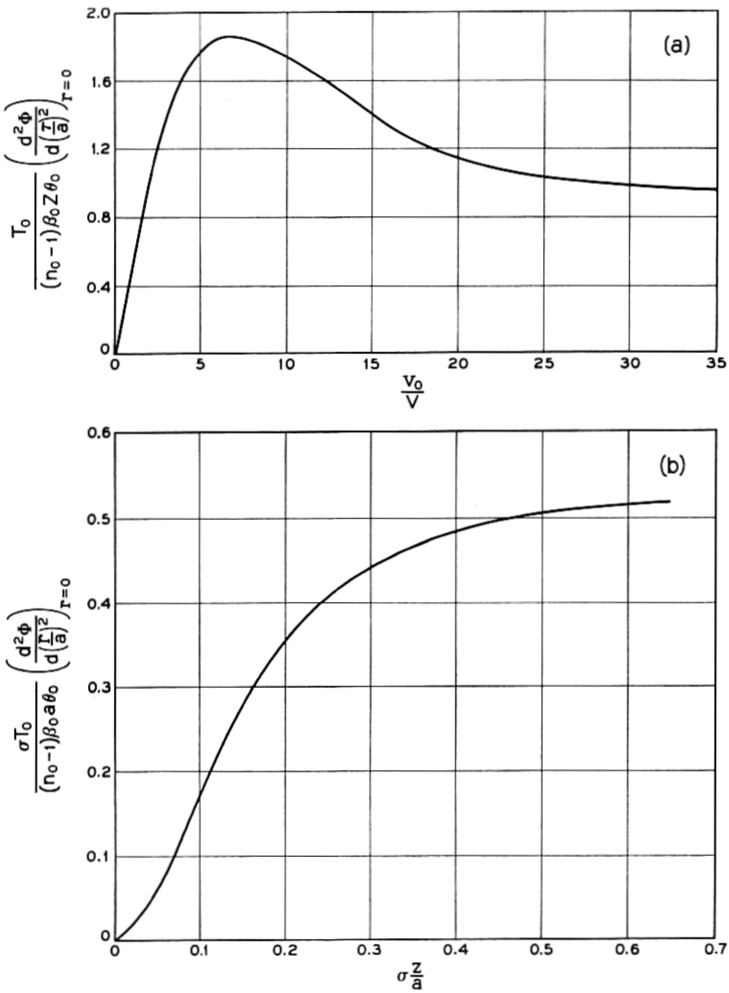

Fig. 8 — (a) Normalized $d^2\Phi/d(r/a)^2$ vs gas velocity; (b) normalized $d^2\Phi/d(r/a)^2$ vs longitudinal position.

gas lens could be treated as a thin lens, it would act very similar to a glass lens with spherically curved surfaces.

However, the gas lens is not thin and the question presents itself: how do different sections of the lens contribute to the over-all focusing effect? Fig. 12 shows the phase difference $\Delta\Phi$ between a ray on the axis at $r = 0$ and a ray at r for a fixed value $v_0/V = 6.9$ for different sections

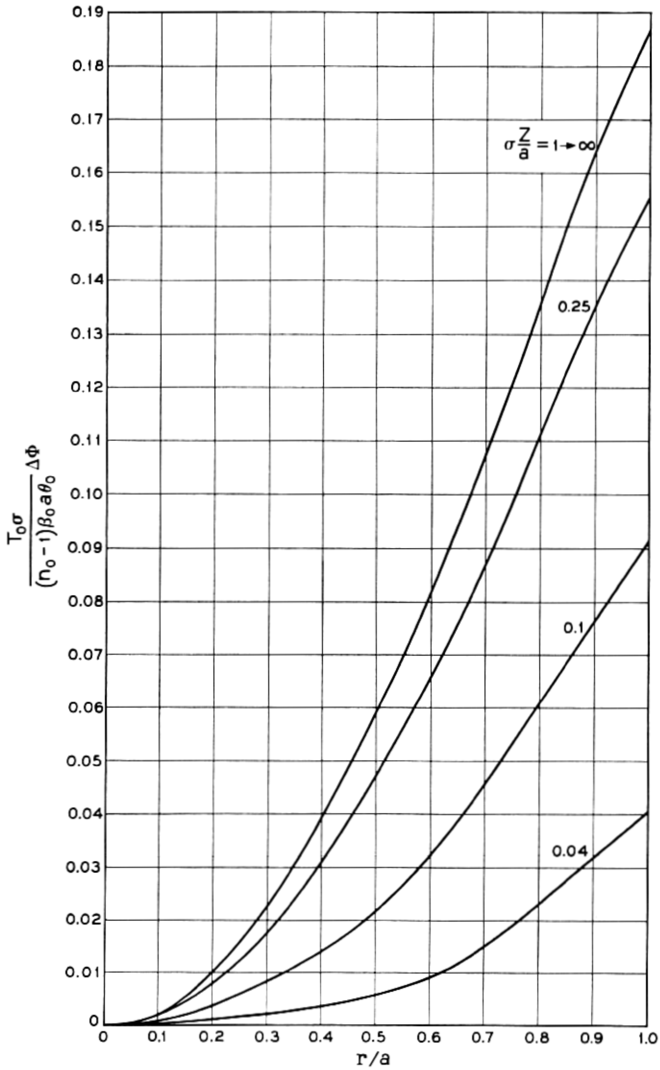

Fig. 9 — Normalized $\Delta\Phi$ vs r/a with longitudinal position as a parameter.

of the lens. The curve showing $\Delta\Phi$ for the section $0 \rightarrow z$ has already been shown in Fig. 11. The other curves show $\Delta\Phi$ in the first $\frac{1}{3}$ of the lens [curve $0 \rightarrow (\frac{1}{3})z$], the second $\frac{1}{3}$[curve $(\frac{1}{3})z \rightarrow (\frac{2}{3})z$], and the last $\frac{1}{3}$[curve $(\frac{2}{3})z \rightarrow z$]. The contributions are surprisingly different at different radii and do not resemble simple $(r/a)^2$ dependences. However, they all add

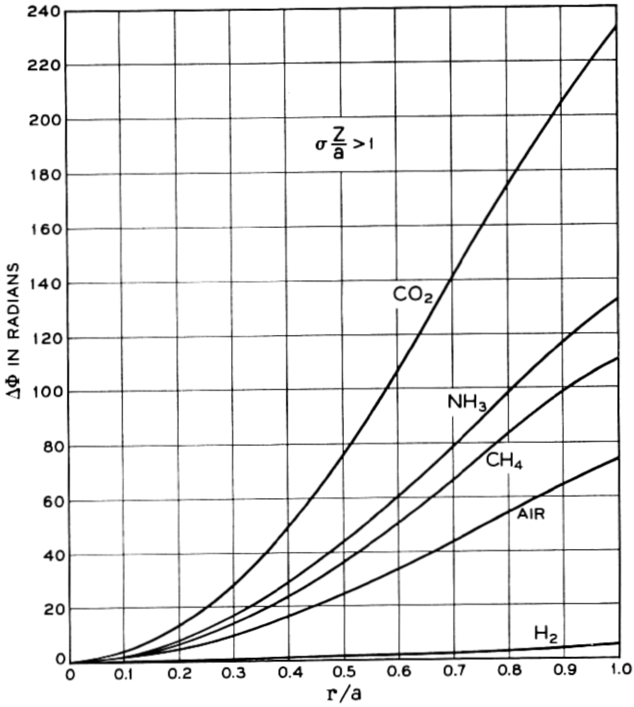

Fig. 10 — $\Delta\Phi$ vs r/a when $a = 0.125$ inch, $v_0 = 212$ cm/sec (corresponding to 2 liters/min), $\lambda_0 = 5890$ Å, $T_W = 343°$K, $T_0 = 293°$K and $\sigma Z/a > 1$.

up to the $0 \rightarrow z$ curve which does resemble the $(r/a)^2$ dependence very closely, as was shown in Fig. 11.

V. FIGURE OF MERIT

The focusing action of the gas lens becomes independent of the length of the lens if $\sigma(z/a) > 1$, as Figs. 6 and 8(b) show. We also know that, for a fixed length of the lens, there is an optimum flow velocity, as shown by Figs. 7 and 8.

For practical applications one would like not only to obtain an effective lens but also to do so with a minimum expenditure of power. It is, therefore, interesting to study the lens action, that is, $d^2\Phi/d(r/a)^2$, per unit of applied power. We may introduce the ratio

$$M = \frac{1}{P}\left(\frac{d^2\Phi}{d(r/a)^2}\right)_{r=0} \tag{21}$$

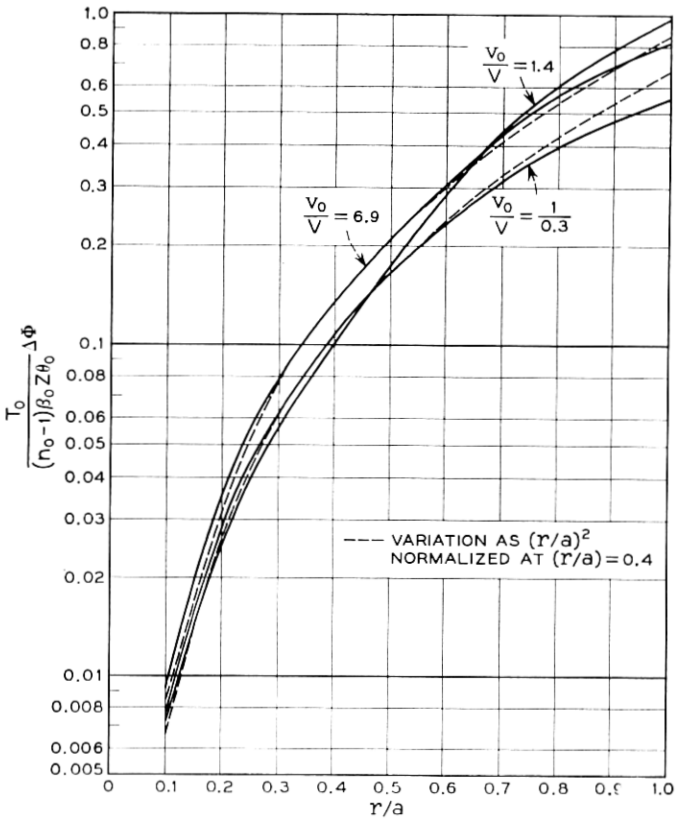

Fig. 11 — Normalized $\Delta\Phi$ vs r/a with gas velocity as a parameter; dotted curve represents variation as $(r/a)^2$, normalized to ordinate at $r/a = 0.4$.

as the figure of merit of the lens. From (6) and (20) we obtain

$$M = \frac{2\beta_0}{\pi T_0} \cdot \frac{n_0 - 1}{k} \left\{ \frac{F}{1 - 0.820 \exp\left(-7.316 \dfrac{V}{v_0}\right)} \right\} \tag{22a}$$

with

$$F = 0.738 \left[1 - \exp\left(-7.316 \frac{V}{v_0}\right)\right] - 0.405 \cdot \left[1 - \exp\left(-44.3 \frac{V}{v_0}\right)\right] + 0.192 \left[1 - \exp\left(-106 \frac{V}{v_0}\right)\right]. \tag{22b}$$

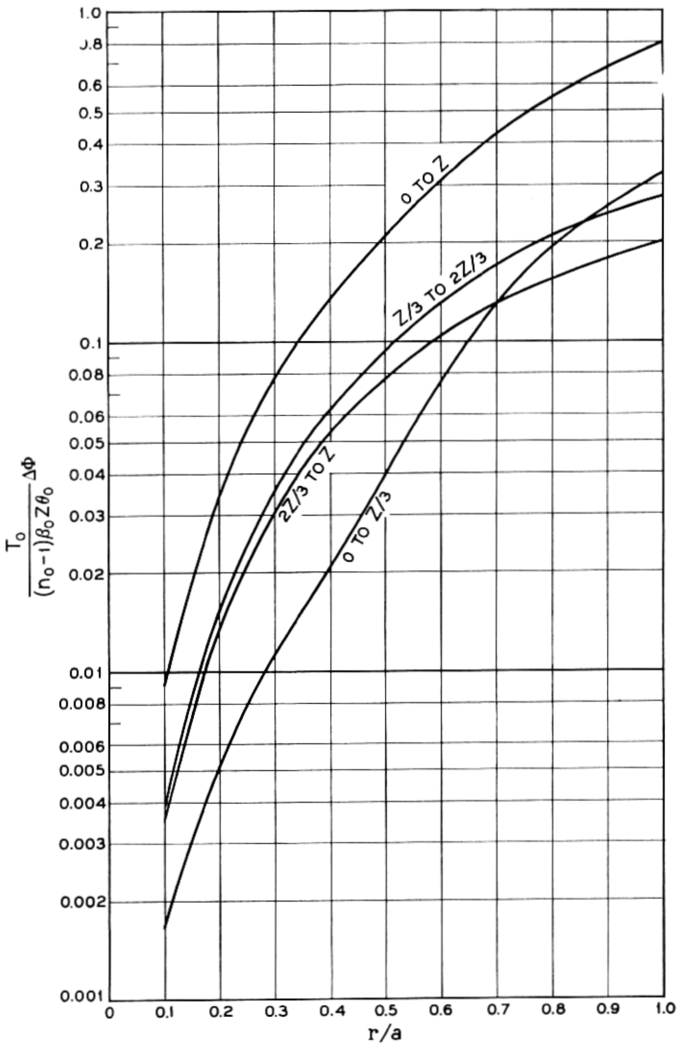

Fig. 12 — $\Delta\Phi$ contributed by first third, second third, and output third of gas lens for $v_0/V = 6.9$.

For a given value of v_0/V, the figure of merit is proportional to $(n_0 - 1)/k$. It is advantageous to make this number as large as possible.

The figure of merit M, given in (22b), is plotted in Fig. 13.

The gases in Table II are arranged in decreasing order of $(n_0 - 1)/k$. Of all the gases listed in that table, carbon dioxide is best suited for a gas

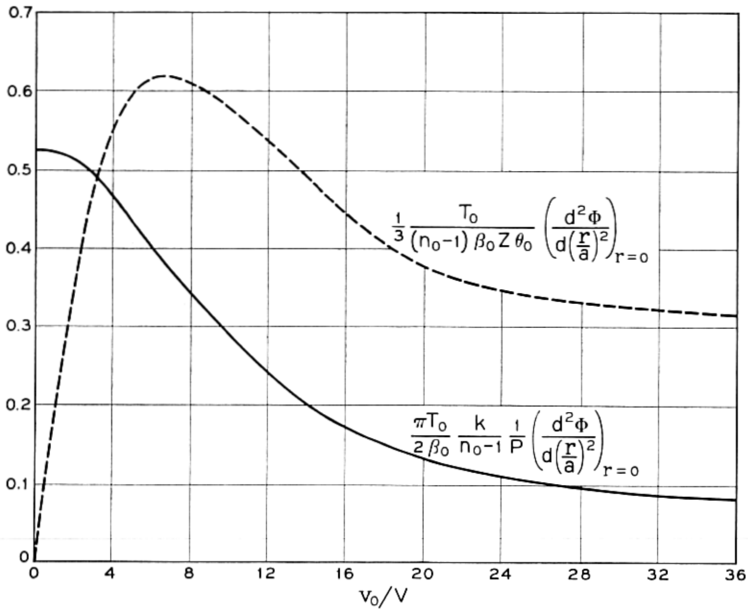

Fig. 13 — Figure of merit given by (22a) and focusing power (as in Fig. 8) vs gas velocity.

lens. This does not mean, however, that gases with larger values of $(n_0 - 1)/k$ cannot be found.

VI. FOCAL LENGTH

We have seen that $\Delta\Phi$ varies nearly as $(r/a)^2$. In the region where the lens is weak, we may treat it as a thin lens and obtain directly a simple expression for focal length.

For any thin lens it may be shown that the focal length f is given by

$$f = \tfrac{1}{2}\beta_0(r^2/\Delta\Phi) \tag{23}$$

where

$\Delta\Phi$ is the phase shift added on axis as compared to that for a ray at radius r

β_0 = phase constant of the region surrounding the lens.

We obtain $\Delta\Phi$ from (17), which is given by the following for the gas velocity set to maximize $\Delta\Phi$ at $(r/a) = 0.4$ — i.e., at $(v_0/V) = 6.9$:

$$\Delta\Phi = 0.839(r/a)^2(\theta_0/T_0)\beta_0 z(n_0 - 1). \tag{24}$$

Putting (24) into (23) we obtain the following expression for the focal length of a weak gas lens:

$$f = 0.596 \frac{\lambda^2}{z} \frac{T_0}{\theta_0(n_0 - 1)}. \tag{25}$$

If $a = 0.125$ inch, $z = 5$ inches, $T_0 = 293°K$, and $\theta_0 = 20°C$, we find $f \cong 5$ feet using CO_2 as the gas and $f \cong 8$ feet using air as the gas; the power transferred to the gas with CO_2 would be 0.0775 cal/sec = 0.325 watt.

When the gas lens is not weak, one should take into account that the refractive index varies both with radial position and with longitudinal position. Work is under way to analyze this very difficult situation. A simpler approach, and one which should give a first-order answer for gas lenses operated near the velocity producing maximum $\Delta\Phi$ [see (17)], is to assume a medium within the lens

$$n(r,x) = n_a(1 - \tfrac{1}{2}a_2r^2) \tag{26}$$

where

x = distance (within lens)from start of lens
r = radius
n_a = index of refraction on the axis.

For the gas velocity $(v_0/V) = 6.9$ it may be shown that

$$a_2 = \frac{1.68}{n_a a^2} \frac{\theta_0}{T_0} (n_0 - 1). \tag{27}$$

In other unpublished work the authors have shown that the radial position of a ray (or of the axis of a Gaussian beam mode) is

$$r = r_i \cos \sqrt{a_2}x + \frac{r_i'}{\sqrt{a_2}} \sin \sqrt{a_2}x \tag{28}$$

where

r_i = displacement of ray at lens input
r_i' = slope of ray at lens input.

We can use this general result to specify the focal length of a strong (or weak) gas lens with reference to Fig. 14. All input rays with zero slope will converge to a point on the axis a distance d beyond the output face of the lens (Fig. 14), where

$$d = \frac{1}{\sqrt{a_2}} \cot (\sqrt{a_2}t), \tag{29}$$

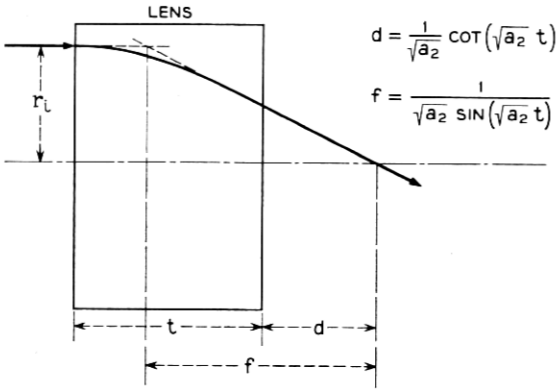

Fig. 14 — Diagram defining focal length and position of the equivalent thin lens for a thick gas lens.

t is the length of the lens, and n_a has been taken as unity. An equivalent thin lens may be located a distance f back from the focal point, where

$$f = \frac{1}{\sqrt{a_2} \sin(\sqrt{a_2}t)} \cdot \tag{30}$$

This expression for focal length is valid up to $(\sqrt{a_2}t) = \pi/2$, at which point $d = 0$. For $(\sqrt{a_2}t) > \pi/2$ the rays cross within the lens, per (28). For $(\sqrt{a_2}\, t) \ll 1$ it may be shown that (30) passes into (25) and the location of the equivalent thin lens is in the center of the distributed lens.

E. A. J. Marcatili[4] has solved Maxwell's equations for a medium characterized by (26) and has found the normal modes for a sequence of lenses composed of segments of such a medium. This work relates closely to a sequence of gas lenses described in this paper. Experiments with tubular thermal gas lenses are reported by A. C. Beck.[5]

VII. CONCLUSION

When a cool gas is blown through a warmer tube, the gas at the axis has a lower temperature than that near the walls. Thus the density and refractive index is larger at the axis and a converging lens is formed. If the tube were at a lower temperature than the input gas, a diverging lens would be formed.

There is an optimum gas velocity for maximizing the focusing power of such lenses, and expressions are given for this velocity. It turns out that the optimum transit time for gas through the tube is approximately the time constant for temperature changes in a gas at rest in the tube, which for typical gases (air and carbon dioxide) is about 0.1 second in a $\frac{1}{4}$-inch ID tube. Although not discussed in this paper, it is found that a $\frac{1}{4}$-inch tube 6 inches long yields (at the optimum velocity for focusing power) a Reynolds number well below that at which turbulence is expected.

Expressions are given for focal length and a figure of merit expressed as focusing power per watt of power transferred to the moving gas.

The best gas is one with a maximum $(n - 1)/k$, where n is the refractive index and k is the heat conductivity.

APPENDIX A

Derivation of (10)

The relation (10)

$$V/v_0 = 0.173(t_0/\tau) \tag{31}$$

can be derived as follows.

The time development of a cool gas resting in a tube of wall temperature T_w can be described as follows

$$T = T_w - 2(T_w - T_0) \sum_{n=1}^{\infty} e^{-\lambda_n t} \frac{J_0\left(w_n \frac{r}{a}\right)}{w_n J_1(w_n)} \tag{32}$$

with

$$\lambda_n = (k/a^2 \rho c_p) w_n^2 \quad \text{and} \quad J_0(w_n) = 0.$$

At $t = 0$, (32) becomes $T(r,0) = T_0$, which is constant throughout the tube's cross section.

As time progresses the exponents $\lambda_n t$ become large, so that very soon the first term of the series is the only contributing factor. Neglecting all the terms except the first, we get

$$\frac{1}{\tau} = \frac{\dfrac{dT(0,t)}{dt}}{T_w - T(0,t)} \cong \lambda_1 = \frac{k}{a^2 \rho c_p} w_1^2 = 5.79 \frac{V}{z} \tag{33}$$

and substituting $z = v_0 t_0$ we get (31). Since we neglected all but the first term in the series, (33) represents the asymptotic value which $1/\tau$ assumes after the initial transients have died down.

To obtain a feeling for the accuracy of the approximation involved in deriving (33), we write down the ratio of the second to the first term in the sum (32) for $r = 0$:

$$\frac{w_1 J_1(w_1)}{w_2 J_1(w_2)} \exp[-(\lambda_2 - \lambda_1)t] = 0.666 \exp\left(-4.26 \frac{t}{\tau}\right).$$

This ratio is 10^{-2} for $t/\tau = 0.986$ and is 10^{-1} for $t/\tau = 0.45$. The approximation is excellent for times $t \geqq \tau$ and is quite good for $t > 0.5\tau$.

For the special example used in Fig. 10 we get for

$$\begin{aligned} CO_2 : \tau &= 0.161 \text{ sec} \\ \text{air:} \quad \tau &= 0.08 \text{ sec.} \end{aligned}$$

APPENDIX B

The Maximum of (17)

We seek an expression for the value of v_0/V which brings $\Delta\Phi$, equation (17), to a maximum:

$$\begin{aligned} \frac{d(\Delta\Phi)}{d(v_0/V)} &= \beta_0 z(n_0 - 1)\frac{\theta_0}{T_0} \\ &\cdot\left[0.202(1 - R_0)\left\{1 - \exp\left(-7.316\frac{V}{v_0}\right)\left(1 + 7.316\frac{V}{v_0}\right)\right\}\right. \\ &- 0.0183(1 - R_1)\left\{1 - \exp\left(-44.3\frac{V}{v_0}\right)\left(1 + 44.3\frac{V}{v_0}\right)\right\} \\ &\left. + 0.00363(1 - R_2)\left\{1 - \exp\left(-106\frac{V}{v_0}\right)\left(1 + 106\frac{V}{v_0}\right)\right\}\right]. \end{aligned} \tag{34}$$

We set (34) equal to zero, and noting that the second and third exponentials are small, we neglect them (to be justified by the solutions thus obtained) yielding

$$\begin{aligned} &\exp\left(-7.316\frac{V}{v_0}\right)\left(1 + 7.316\frac{V}{v_0}\right) \\ &\qquad = 1 - 0.0906\frac{(1 - R_1)}{(1 - R_0)} + 0.018\frac{(1 - R_2)}{(1 - R_0)}. \end{aligned} \tag{35}$$

Equation (35) gives the approximate value of v_0/V at the maximum of $\Delta\Phi$ for any chosen radius, r/a. Additional terms in (34) can be taken if more accuracy is desired, which was done in computing the v_0/V for $r/a = 0.2$, as given in the body of the paper.

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